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Several observations on symplectic, Hamiltonian, and skew-Hamiltonian matrices

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Abstract

We prove a Hamiltonian/skew-Hamiltonian version of the classical theorem relating strict equivalence and T-congruence between pencils of complex symmetric or skew-symmetric matrices. Then, we give a pure symplectic variant of the recent result of Xu concerning the singular value decomposition of a conjugate symplectic matrix. Finally, we discuss implications that can be derived from Veselić's result on definite pairs of Hermitian matrices for the skew-Hamiltonian situation.

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1. Introduction

Let $M_n(\mathbf{C})$ be the set of complex $n \times n$ matrices and M a fixed nonsingular matrix in $M_n(\mathbf{C})$. There are two ways for introducing the scalar product based on M in the space $\mathbf{C}^n$, whose elements are conceived as column vectors. On the one hand, we can define the scalar product as a bilinear form, setting

$$(x, y)_M = x^{\mathrm{T}} M y. \tag{1}$$

Another possibility is to make the scalar product a sesquilinear form by defining

$$(x, y)_M = x^* M y. \tag{2}$$

In the well-known case $M = I_n$ the first choice corresponds to $\mathbf{C}^n$ interpreted as a complex Euclidean space; the second, to the unitary space $\mathbf{C}^n$. As in this particular example, the scalar product is usually required to be symmetric or skew-symmetric.

Consider a matrix $A \in M_n(\mathbf{C})$ as a linear operator acting on $\mathbf{C}^n$. Having the scalar product, we can define the adjoint matrix $A^\star$ by the familiar relation

$$\langle Ax, y\rangle_M = \langle x, A^\star y\rangle_M \quad \forall x, y \in \mathbf{C}^n.$$

It is easy to show that

$$A^\star = \begin{cases} M^{-1}A^{\mathrm{T}}M & \text{in case (1)}, \\ M^{-1}A^*M & \text{in case (2)}. \end{cases} \tag{3}$$

Now, special classes of matrices with respect to the scalar product at hand can be defined in the conventional way:

$$M\text{-orthogonal matrices:} \quad A^\star A = I_n, \tag{4}$$

$$M\text{-symmetric matrices:} \quad A^\star = A, \tag{5}$$

$$M\text{-skew-symmetric matrices:} \quad A^\star = -A. \tag{6}$$

Using (3), one can rewrite these definitions as follows:

$$A^{\mathrm{T}}MA = M, \tag{7}$$

$$A^{\mathrm{T}}M = MA, \tag{8}$$

$$A^{\mathrm{T}}M = -MA, \tag{9}$$

in case (1), and

$$A^*MA = M, \tag{10}$$

$$A^*M = MA, \tag{11}$$

$$A^*M = -MA, \tag{12}$$

in case (2). Relations (8) and (9) say that, for the bilinear scalar product (1), A is M-symmetric (M-skew-symmetric) iff MA is symmetric (resp., skew-symmetric) in the conventional sense. A similar interpretation holds for relations (11) and (12).

In this paper, we are mainly interested in the case when $n = 2m$ and

$$M = J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}. \tag{13}$$

Then, there are special terms for matrix classes defined in (7)–(9), namely, (complex) symplectic, skew-Hamiltonian, and Hamiltonian matrices. Very often, the same terms are used for matrices defined by (10)–(12) with $M = J$. Since we are going to use both types of scalar products, the terminology advocated in [5] (that is, conjugate symplectic, J-skew-Hermitian, and J-Hermitian matrices) will be employed in the latter case.

Our purpose in this paper is to discuss the three facts concerning symplectic, Hamiltonian, and skew-Hamiltonian matrices. In Section 2, we state and prove a Hamiltonian analogue of the following classical theorem (see [1, Chapter XII, Theorem 6]):

Theorem 1. *Let $A + \lambda B$ and $C + \lambda D$ be two strictly equivalent pencils of complex symmetric (or skew-symmetric) matrices; that is, there exist nonsingular matrices P and Q such that*

$$C = PAQ, \quad D = PBQ. \tag{14}$$

Then, the pencils $A + \lambda B$ and $C + \lambda D$ are congruent; i.e., there exists a nonsingular matrix W such that

$$C = W^{\mathrm{T}}AW, \quad D = W^{\mathrm{T}}BW. \tag{15}$$

Theorem 1 implies that the invariants of a symmetric (or skew-symmetric) pencil are the same for congruence transformations as they are for strictly equivalent transformations. (Note that a similar assertion on Hermitian pencils and Hermitian congruence is false.) Our Hamiltonian version of Theorem 1 admits a similar interpretation.

In Section 3, we discuss the following fact, recently discovered in [7]:

Theorem 2. *Every conjugate symplectic matrix $S \in M_{2m}(\mathbf{C})$ admits a singular value decomposition*

$$S = U\Sigma V^* \tag{16}$$

where all of three factors on the right-hand side are conjugate symplectic. In particular,

$$\Sigma = \begin{pmatrix} \Omega & 0 \\ 0 & \Omega^{-1} \end{pmatrix}, \tag{17}$$

where

$$\Omega = \mathrm{diag}(\omega_1, \ldots, \omega_m), \quad \omega_1 \geqslant \cdots \geqslant \omega_m \geqslant 1. \tag{18}$$

We show that a similar assertion holds for symplectic (rather than conjugate symplectic) matrices. Our proof is different from the computational proof in [7] and is more geometrical in its spirit.

Note that, unlike Ω, the singular values in matrix Σ (17) are not in decreasing order as otherwise the symplectic structure of Σ would be destroyed. The same is true of our purely symplectic version of Theorem 2. We also note that Proposition 3, which is used in the proof of this theorem and which concerns the eigenvalue decomposition of a Hermitian symplectic matrix, may be of independent interest.

In Section 4, we derive an implication related to J-skew-Hermitian matrices of the following remarkable fact, found by Veselić in [6]:

Theorem 3. *Let (A, B) be a definite pair of $n \times n$ Hermitian matrices. Let X be a nonsingular matrix that brings (A, B) to the diagonal form (Λ, G), where*

$$G = I_p \oplus (-I_q), \quad p + q = n, \tag{19}$$

by the $*$-congruent transformation, i.e.,

$$X^* A X = \Lambda, \quad X^* B X = G. \tag{20}$$

Then, all of such matrices X have the same condition number with respect to the 2-norm or the Frobenius norm.

Recall that a pair (A, B) is said to be definite if

$$(Ax, x)^2 + (Bx, x)^2 \neq 0 \quad \forall x \neq 0, x \in \mathbf{C}^n.$$

This is equivalent to saying that a certain linear combination of A and B is positive definite.

2. Pencils of Hamiltonian/skew-Hamiltonian matrices

We begin by reminding the reader of the proof of Theorem 1. This will let us see how the assertion of the theorem can be extended to a certain extent.

Proof of Theorem 1. Applying the transpose operation to relations (14), we obtain

$$C = Q^{\mathrm{T}} A P^{\mathrm{T}}, \quad D = Q^{\mathrm{T}} B P^{\mathrm{T}}. \tag{21}$$

Set

$$U = Q P^{-\mathrm{T}}. \tag{22}$$

Then, the left-hand relations in (14) and (21) imply that

$$AU = U^{\mathrm{T}} A. \tag{23}$$

From (23), we easily derive that

$$AU^k = (U^k)^{\mathrm{T}} A, \quad k = 0, 1, 2, \ldots,$$

and, more generally,

$$AS = S^{\mathrm{T}} A \tag{24}$$

for any polynomial S in the matrix U:

$$S = f(U). \tag{25}$$

Assume that S is nonsingular. Then, (24) can be rewritten as

$$A = S^{\mathrm{T}} A S^{-1}.$$

Substituting this in the first relation in (14), we have

$$C = PS^{\mathrm{T}} A S^{-1} Q. \tag{26}$$

Working in the same way with the right-hand relations in (14) and (21), we obtain

$$D = PS^{\mathrm{T}} B S^{-1} Q \tag{27}$$

for the same matrices S. For equalities (26) and (27) to be congruences, it suffices that

$$(PS^{\mathrm{T}})^{\mathrm{T}} = S^{-1} Q,$$

or

$$S^2 = QP^{-\mathrm{T}} = U.$$

Thus, if we choose S as a square root of the matrix U that is a polynomial in U (see [4] for a discussion on square roots of a matrix that are polynomials in that matrix), then (26) and (27) become the desired congruence relations (15) with

$$W = S^{-1} Q = (\sqrt{U})^{-1} Q. \qquad \square$$

Remark. To derive (26), we only need that both A and C be symmetric (or skew-symmetric). The same applies to relation (27) and the matrices B and D. However, nothing changes in the proof above if A and C are symmetric matrices, while B and D are skew-symmetric, or vice versa. Therefore, we obtain the following amendment to Theorem 1:

Theorem 1′. *Let $A + \lambda B$ and $C + \lambda D$ be two strictly equivalent pencils of complex matrices with A and C being symmetric, while B and D are skew-symmetric, or vice versa. Then, the pencils $A + \lambda B$ and $C + \lambda D$ are congruent.*

To state a version of Theorem 1 for Hamiltonian/skew-Hamiltonian matrices, we should first define what is the congruence transformation for these matrices. We adopt the following

Definition. Let A and C be matrices in $M_{2m}(\mathbf{C})$. We say that A and C are J-congruent if there exists a nonsingular matrix W such that

$$C = W^{\star} A W,$$

where the adjoint matrix is to be understood in the sense of the upper formula in (3) with $M = J$ (see (13)).

It is easy to verify that J-congruence transformations preserve the property of a matrix to be Hamiltonian, skew-Hamiltonian, or symplectic.

Now, we are in a position to formulate the J-version of Theorems 1 and 1′.

Theorem 4. *Let A, B, C, D be matrices in $M_{2m}(\mathbf{C})$ such that*

(i) *all four matrices are Hamiltonian; or*

(ii) *all four matrices are skew-Hamiltonian; or*

(iii) *A and C are Hamiltonian, while B and D are skew-Hamiltonian, or vice versa.*

Then, if the pencils $A + \lambda B$ and $C + \lambda D$ are strictly equivalent, then they are J-congruent.

Proof. Define

$$\tilde{A} = JA, \quad \tilde{B} = JB, \quad \tilde{C} = JC, \quad \tilde{D} = JD.$$

The fact that the pencils $A + \lambda B$ and $C + \lambda D$ are strictly equivalent entails the equivalence relation between the pencils $\tilde{A} + \lambda\tilde{B}$ and $\tilde{C} + \lambda\tilde{D}$. Indeed, relations (14) imply that

$$\tilde{C} = (JPJ^{-1})\tilde{A}Q \quad \text{and} \quad \tilde{D} = (JPJ^{-1})\tilde{B}Q.$$

Now, the pencils $\tilde{A} + \lambda\tilde{B}$ and $\tilde{C} + \lambda\tilde{D}$ are either skew-symmetric or symmetric, or matrices $\tilde{A}$ and $\tilde{C}$ are skew-symmetric, while $\tilde{B}$ and $\tilde{D}$ are symmetric, or vice versa. By Theorems 1 and 1′, these pencils are congruent, i.e., there exists a nonsingular matrix W such that

$$\tilde{C} = W^{\mathrm{T}}\tilde{A}W \quad \text{and} \quad \tilde{D} = W^{\mathrm{T}}\tilde{B}W,$$

or

$$JC = W^{\mathrm{T}}JAW \quad \text{and} \quad JD = W^{\mathrm{T}}JBW,$$

or

$$C = (J^{-1}W^{\mathrm{T}}J)AW \quad \text{and} \quad D = (J^{-1}W^{\mathrm{T}}J)BW.$$

These are the desired relations

$$C = W^{\star}AW \text{ and } D = W^{\star}BW. \qquad \square$$

Remark. With obvious modifications, Theorem 4 can be extended to symmetric or skew-symmetric pencils with respect to many other scalar products. In particular, this is true for the important case of a pseudo-Euclidean scalar product, where the matrix M in (1) is

$$I_p \oplus (-I_p), \quad p, q > 0, \quad p + q = n.$$

3. The SVD of a symplectic matrix

In this section, we prove a symplectic counterpart of Theorem 2. Recall that the latter deals with the SVD of a conjugate symplectic matrix.

Theorem 5. *Every symplectic matrix $S \in M_{2m}(\mathbf{C})$ admits a singular value decomposition of form (16), where all of the three factors U, Σ and V are symplectic. The diagonal matrix Σ is the same as in (17) and (18).*

We precede the proof of Theorem 5 by a number of preparatory propositions. The first of them is well known, being a simple consequence of the fact that the inverse of a nonsingular matrix A is a polynomial in A.

Proposition 1. *Let $A \in M_n(\mathbf{C})$ be a nonsingular matrix such that A^{-1} is similar to A. Then, if $\lambda \neq \pm 1$ is an eigenvalue of A, then $\frac{1}{\lambda}$ also is an eigenvalue. Moreover, A has the same Jordan structure for $\frac{1}{\lambda}$ as it has for λ.*

Corollary 1. *If $\lambda \neq \pm 1$ is an eigenvalue of a symplectic matrix S, then $\frac{1}{\lambda}$ also is an eigenvalue and the Jordan structures associated with λ and $\frac{1}{\lambda}$ are identical.*

Proof. It follows from the definition of a symplectic matrix (see (7) with $M = J$) that

$$S^{-1} = J^{-1} S^{\mathrm{T}} J;$$

i.e., S^{-1} is similar to S^{T}. On the other hand, S^{T} is always similar to S (see, e.g., [3, Section 3.2.3]). Hence, Proposition 1 applies to S. $\square$

Remark. Obviously, an assertion similar to Corollary 1 holds for an M-orthogonal matrix A whatever is the matrix M.

Proposition 2. *Let S be a symplectic matrix. Then, $\bar{S}$ and S^{T} also are symplectic.*

Proof. To prove the first assertion, just take element-wise conjugation for both sides of the relation

$$S^{\mathrm{T}} J S = J. \tag{28}$$

For proving the second assertion, invert both sides in (28):

$$S^{-1} J S^{-\mathrm{T}} = J. \tag{29}$$

Here, we used the relation $J^{-1} = -J$. Now, rewrite (29) as

$$J = S J S^{\mathrm{T}} = (S^{\mathrm{T}})^{\mathrm{T}} J (S^{\mathrm{T}}),$$

which shows that S^{T} is symplectic. $\square$

Corollary 2. *If S is symplectic, then:*
(a) *S^* is symplectic;*
(b) *S^*S and SS^* are symplectic;*
(c) *if $\sigma \neq 1$ is a singular value of S, then $\frac{1}{\sigma}$ is also a singular value; moreover, σ and $\frac{1}{\sigma}$ have identical multiplicities.*

Proof. Assertion (a) is an immediate implication of Proposition 2. Assertion (b) follows from (a) and the fact that the set $\mathscr{S}_{2m}$ of symplectic matrices of a fixed order $2m$ is a multiplicative group. Finally, assertion (c) follows from Proposition 1 applied to the symplectic (and Hermitian positive definite) matrix S^*S (or SS^*). $\square$

Proposition 3. *Let $H \in M_{2m}(\mathbf{C})$ be Hermitian and symplectic at the same time. Then, H admits a symplectic eigenvalue decomposition (EVD)*

$$H = V\Lambda V^*, \tag{30}$$

where the diagonal matrix Λ and the unitary matrix V are symplectic.

Proof. Let λ be an eigenvalue of H and v be an associated unit eigenvector:

$$Hv = \lambda v.$$

Since H is symplectic, we have

$$H^{\mathrm{T}}JH = J.$$

It follows that

$$Jv = H^{\mathrm{T}}JHv = \lambda H^{\mathrm{T}}Jv$$

or

$$H^{\mathrm{T}}(Jv) = \frac{1}{\lambda}(Jv).$$

Taking element-wise conjugation for both sides of this relation and making use of the fact that H is Hermitian, we obtain

$$H(J\bar{v}) = \frac{1}{\lambda}(J\bar{v}).$$

Thus, $J\bar{v}$ is a unit eigenvector of H associated with the eigenvalue $\frac{1}{\lambda}$. If λ is distinct from 1 and -1, then λ and $\frac{1}{\lambda}$ are different; hence, the corresponding eigenvectors v and $J\bar{v}$ must be orthogonal:

$$v^*J\bar{v} = 0. \tag{31}$$

Suppose that exactly r eigenvalues of H have moduli greater than 1. Let

$$\lambda_1, \lambda_2, \ldots, \lambda_r$$

be these eigenvalues and

$$v_1, v_2, \ldots, v_r \tag{32}$$

be the corresponding orthonormal set of eigenvectors. Then,

$$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_r}$$

are the eigenvalues of H whose moduli are less than 1, and

$$J\bar{v}_1, J\bar{v}_2, \ldots, J\bar{v}_r \tag{33}$$

are the corresponding orthonormal eigenvectors. Moreover, the sets (32) and (33) are orthogonal to each other because they correspond to non-overlapping groups of eigenvalues of the Hermitian matrix H.

We first assume that $r = m$. Then, we construct a matrix from vectors (32) and (33), namely,

$$V = \begin{pmatrix} v_1 & \cdots & v_m & -J\bar{v}_1 & \cdots & -J\bar{v}_m \end{pmatrix}. \tag{34}$$

It is clear from what was said above that V is a unitary matrix. Since the columns of V are the eigenvectors of H, we have equality (30) with

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_1^{-1} \end{pmatrix}, \tag{35}$$

where

$$\Lambda_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_m).$$

It is obvious that Λ is a symplectic matrix. It remains to verify that V is symplectic as well. To this end, we set

$$W = (v_1 \quad \cdots \quad v_m)$$

and rewrite V as

$$V = \begin{pmatrix} W & -J\overline{W} \end{pmatrix}.$$

Now, we have

$$V^{\mathrm{T}} J V = \begin{pmatrix} W^{\mathrm{T}} \\ W^* J \end{pmatrix} \begin{pmatrix} JW & \overline{W} \end{pmatrix} = \begin{pmatrix} W^{\mathrm{T}} J W & W^{\mathrm{T}} \overline{W} \\ -W^* W & W^* J \overline{W} \end{pmatrix}.$$

The orthonormality of set (32) implies that

$$W^* W = (v_k^* v_l)_{k,l=1}^m = I_m$$

and

$$W^{\mathrm{T}} \overline{W} = \overline{W^* W} = I_m.$$

The orthogonality between the sets (32) and (33) amounts to the matrix relation

$$W^* J \overline{W} = 0.$$

Therefore,

$$W^{\mathrm{T}} J W = \overline{W^* J \overline{W}} = 0.$$

It follows that

$$V^{\mathrm{T}} J V = J,$$

which completes the proof in the case $r = m$.

Now, we consider the case $r < m$. We first show that each of the eigenvalues -1 and 1 has an even multiplicity. Since the sum of the two multiplicities is the even number

$$2m - 2r,$$

both multiplicities have the same parity. Thus, it suffices to prove that the multiplicity s of $\lambda = 1$ is even.

Let

$$z_1, \ldots, z_s$$

be a complete set of eigenvectors associated with $\lambda = 1$. Then,

$$J\bar{z}_1, \ldots, J\bar{z}_s$$

is another complete set of eigenvectors for $\lambda = 1$. We set

$$Z = (z_1 \ \ldots \ z_s).$$

Since the columns of Z and $J\bar{Z}$ are two bases in the same subspace, the matrix

$$K = Z^* J \overline{Z}$$

must be nonsingular. It is easy to see that K is skew-symmetric; hence, its dimension s must be an even number.

Recall that

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r \tag{36}$$

are the eigenvalues of H whose moduli are greater than 1 and

$$v_1, v_2, \ldots, v_r \tag{37}$$

are the orthonormal eigenvectors associated with the eigenvalues in (36). Moreover,

$$J\bar{v}_1, J\bar{v}_2, \ldots, J\bar{v}_r \tag{38}$$

is the orthonormal set of eigenvectors of H associated with the eigenvalues

$$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_r}.$$

We show that sets (37) and (38) can be complemented by the appropriate set of eigenvectors for $\lambda = 1$ and $\lambda = -1$.

Let $\mathscr{L}$ be the eigenspace for $\lambda = 1$. It turns out that an orthonormal basis in $\mathscr{L}$ can be composed of pairs of the form

$$v, J\bar{v}.$$

Indeed, take an arbitrary unit vector $v \in \mathscr{L}$; then, $J\bar{v}$ also is an eigenvector for the same eigenvalue $\lambda = 1$. Moreover, we still have the orthogonality relation (31):

$$v^* J\bar{v} = \bar{v}^{\mathrm{T}} J\bar{v} = \overline{v^{\mathrm{T}} J v} = 0,$$

just because J is a skew-symmetric matrix.

Now, we set

$$v_{r+1} = v, \quad \mathscr{M}_1 = \operatorname{span}\{v_{r+1}, J\bar{v}_{r+1}\}.$$

If $s > 2$, then we apply the orthogonal decomposition

$$\mathscr{L} = \mathscr{M}_1 \oplus \mathscr{L}_1, \quad \mathscr{L}_1 = \mathscr{M}_1^{\perp}.$$

At the second step, take an arbitrary unit vector v_{r+2} in $\mathscr{L}_1$ and set

$$\mathscr{M}_2 = \operatorname{span}\{v_{r+2}, J\bar{v}_{r+2}\}.$$

If $s > 4$, then apply the orthogonal decomposition again:

$$\mathscr{L}_1 = \mathscr{M}_2 \oplus \mathscr{L}_2, \quad \mathscr{L}_2 = \mathscr{M}_2^{\perp}.$$

Then, take an arbitrary unit vector v_{r+3} in $\mathscr{L}_2$, and so on. As a result, the orthonormal vectors

$$v_{r+1}, \ldots, v_t$$

will be constructed, which can be adjoined to set (37). The set

$$J\bar{v}_{r+1}, \ldots, J\bar{v}_t$$

can then be adjoined to (38).

If $t < m$, then the same construction is applied to the eigenspace for $\lambda = -1$. It results in the orthonormal vectors

$$v_{t+1}, \ldots, v_m$$

and

$$J\bar{v}_{t+1}, \ldots, J\bar{v}_m,$$

which are adjoined to sets (37) and (38), respectively.

As soon as the augmented systems

$$v_1, \ldots, v_m$$

and

$$J\bar{v}_1, \ldots, J\bar{v}_m$$

have been obtained, the rest of the proof is a word-for-word repetition of the proof in the case $r = m$. □

We are now in a position to prove Theorem 5.

Proof of Theorem 5. As S is symplectic, by Corollary 2, $H = S^*S$ is a symplectic matrix, hence, Proposition 3 can be applied.

Let (30) be the symplectic EVD of H.

$$H = V\Lambda V^*,$$

with the symplectic matrix V and Λ as in (35). Set

$$\Omega = \Lambda_1^{1/2} = \operatorname{diag}\left(\lambda_1^{1/2}, \ldots, \lambda_m^{1/2}\right) = \operatorname{diag}(\sigma_1, \ldots, \sigma_m)$$

and

$$\Sigma = \begin{pmatrix} \Omega & 0 \\ 0 & \Omega^{-1} \end{pmatrix}. \tag{39}$$

Then

$$\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_m \geqslant 1$$

and the matrix in (39) is obviously symplectic. Finally, define

$$U = SV\Sigma^{-1}$$

As all of the three matrices on the right-hand side are symplectic, Theorem 4 is proved. □

To conclude this section, we mention the following beautiful fact given in the recent paper [2]:

Proposition 4. *Let M in (3) be a unitary matrix. Then, for any M-orthogonal matrix A, both factors of its polar decomposition,*

$$A = UH, \tag{40}$$

are themselves M-orthogonal.

The authors of [2] say nothing of the other polar decomposition

$$A = H_1U, \tag{41}$$

where

$$H_1 = (AA^*)^{1/2},$$

although, obviously, Proposition 4 holds true for (41). We could have based the proof of Theorem 5 on using Proposition 4 with $M = J$.

4. A *J*-skew-Hermitian implication of Veselić's result

Define

$$H = \mathrm{i}J, \tag{42}$$

where J is the matrix in (13). Note that both positive and negative inertia indices of the Hermitian matrix H are equal to m.

Let A be a J-skew-Hermitian matrix such that the Hermitian pair (HA, H) is definite. One situation where this assumption is fulfilled is discussed at the end of this section. Let X be any nonsingular matrix that brings the pair (HA, H) through the $*$-congruent transformation to the diagonal form (Λ, G):

$$X^*(HA)X = \Lambda, \qquad X^*HX = G, \tag{43}$$

and

$$G = I_m \oplus (-I_m).$$

By Theorem 3, all of such matrices X have the same spectral (or Frobenius) condition number.

It follows from relations (43) that

$$X^*H = GX^{-1}$$

and

$$GX^{-1}AX = \Lambda.$$

Setting

$$\Lambda = \Lambda_1 \oplus \Lambda_2,$$

where Λ_1 and Λ_2 are $m \times m$ diagonal matrices with real diagonal entries, we obtain

$$X^{-1}AX = G\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & -\Lambda_2 \end{pmatrix}.$$

Thus, X brings A to diagonal form through the similarity transformation; moreover, X satisfies the second relation in (43).

Define

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} I_m & \mathrm{i}I_m \\ -\mathrm{i}I_m & -I_m \end{pmatrix}.$$

It is easy to verify that $P^* = P$ and

$$P^*HP = PHP = G. \tag{44}$$

Setting

$$Y = XP,$$

we find from (43) and (44) that

$$Y^*HY = H,$$

i.e., Y is a conjugate symplectic matrix. Moreover,

$$Y^{-1}AY = P(X^{-1}AX)P = P(G\Lambda)P = \begin{pmatrix} \Lambda_1 - \Lambda_2 & \mathrm{i}(\Lambda_1 + \Lambda_2) \\ -\mathrm{i}(\Lambda_1 + \Lambda_2) & \Lambda_1 - \Lambda_2 \end{pmatrix}.$$

We can sum up the situation by saying that Y is a conjugate symplectic matrix that brings A through the similarity transformation to the special form

$$\begin{pmatrix} D_1 & \mathrm{i}D_2 \\ -\mathrm{i}D_2 & D_1 \end{pmatrix}, \tag{45}$$

where D_1 and D_2 are real diagonal $m \times m$ matrices.

From (45), the spectrum of A can easily be found. Indeed, it is given by the diagonal entries of the matrices

$$\Lambda_1 = D_1 + D_2 \quad \text{and} \quad \Lambda_2 = D_2 - D_1.$$

Since $\|Y\|_p = \|XP\|_p = \|X\|_p$ for $p = 2, F$, we have by Vecelić's result

Corollary 3. *Let A be a J-skew-Hermitian matrix such that the Hermitian pair $(\mathrm{i}JA, \mathrm{i}J)$ is definite. All conjugate symplectic matrices Y which bring A through similarity transformations to the special form* (45) *have the same spectral* (*or Frobenius*) *condition number*.

Remark. Let A be a J-skew-Hermitian matrix of order $n = 2m$. Then, being partitioned into $m \times m$ blocks, A has the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{11}^* \end{pmatrix}, \tag{46}$$

where A_{12} and A_{21} are skew-Hermitian matrices. The matrix $B = \mathrm{i}A$ is J-Hermitian. Its block form corresponding to (46) is

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & -B_{11}^* \end{pmatrix}, \tag{47}$$

where $B_{12} = \mathrm{i}A_{12}$ and $B_{21} = \mathrm{i}A_{21}$ are Hermitian matrices. The pair $(HA,\ H)$ will obviously be definite if the matrix HA is definite. A necessary condition for this to happen is that one of the blocks B_{12} and B_{21} in (47) is positive definite, while the other is negative definite.

5. Conclusions

In this paper, we have proved the following three statements:

(1) Let A, B, C, D be matrices in $M_{2m}(\mathbf{C})$ such that
   (i) all four matrices are Hamiltonian; or
   (ii) all four matrices are skew-Hamiltonian; or
   (iii) A and C are Hamiltonian, while B and D are skew-Hamiltonian, or vice versa.
   Then, if the pencils $A + \lambda B$ and $C + \lambda D$ are strictly equivalent, then they are J-congruent.

(2) Every symplectic matrix $S \in M_{2m}(\mathbf{C})$ admits a singular value decomposition, where all of the three factors U, Σ and V are symplectic.
(3) Let A be a J-skew-Hermitian matrix such that the Hermitian pair $(\mathrm{i}JA, \mathrm{i}J)$ is definite. All conjugate symplectic matrices Y which bring A through similarity transformations to the special form (45) have the same spectral (or Frobenius) condition number.

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